

Elliptic parametrization of Pfaff integrable hierarchies in the zero dispersion limit

V. Akhmedova* A. Zabrodin †

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Abstract

We show that the dispersionless limits of the Pfaff-KP (also known as the DKP or Pfaff lattice) and the Pfaff-Toda hierarchies admit a reformulation through elliptic functions. In the elliptic form they look like natural elliptic deformations of the dispersionless KP and 2D Toda hierarchy respectively.

1 Introduction

In this paper we consider dispersionless limits of the Pfaff-KP and Pfaff-Toda hierarchies. The aim of the paper is to present their reformulation in terms of elliptic functions. In this form they look like natural “elliptic deformations” of the usual Kadomtsev-Petviashvili (KP) and 2D Toda lattice (2DTL) hierarchies in the zero dispersion limit.

The Pfaff-KP hierarchy (also known as DKP, coupled KP, Pfaff lattice) is one of the integrable hierarchies with D_∞ symmetries introduced by Jimbo and Miwa in 1983 [1]. Since then it emerged under different names in different contexts [2]–[8]. Its algebraic structure and some particular solutions were studied in [9, 10, 11]. The term “Pfaff” is due to the fact that soliton-like solutions are expressed through Pfaffians. In this paper we will refer to this hierarchy as the Pfaff-KP one. Bearing certain similarities with the modified KP and Toda chain hierarchies, it is essentially different and worse understood.

The 2D Pfaff-Toda hierarchy suggested in [12, 13] is an extension of the Pfaff-KP hierarchy which relates to it in the same way as the 2DTL relates to the KP hierarchy. In particular, the extension Pfaff-KP \rightarrow Pfaff-Toda implies doubling of the set of hierarchical times. Here we deal with “real forms” of the hierarchies which means that the

*Laboratory of Mathematical Physics, National Research University Higher School of Economics, 20 Myasnitskaya Ulitsa, Moscow 101000, Russia, e-mail: valeria-58@yandex.ru

†Institute of Biochemical Physics, 4 Kosygina st., Moscow 119334, Russia; ITEP, 25 B. Cheremushkinskaya, Moscow 117218, Russia and Laboratory of Mathematical Physics, National Research University Higher School of Economics, 20 Myasnitskaya Ulitsa, Moscow 101000, Russia, e-mail: zabrodin@itep.ru

KP times are assumed to be real while the two sets of Toda times are complex conjugate to each other.

The dispersionless version of the Pfaff-KP hierarchy (which we abbreviate as dPfaff-KP) was suggested in [13, 14]. In the Hirota form, it is an infinite system of differential equations

$$e^{D(z)D(\zeta)F} \left(1 - \frac{1}{z^2\zeta^2} e^{2\partial_{t_0}(2\partial_{t_0}+D(z)+D(\zeta))F} \right) = 1 - \frac{\partial_{t_1}D(z)F - \partial_{t_1}D(\zeta)F}{z - \zeta} \quad (1)$$

$$e^{-D(z)D(\zeta)F} \frac{z^2 e^{-2\partial_{t_0}D(z)F} - \zeta^2 e^{-2\partial_{t_0}D(\zeta)F}}{z - \zeta} = z + \zeta - \partial_{t_1}(2\partial_{t_0} + D(z) + D(\zeta))F \quad (2)$$

for the function $F = F(\mathbf{t})$ of the infinite number of (real) times $\mathbf{t} = \{t_0, t_1, t_2, \dots\}$, where

$$D(z) = \sum_{k \geq 1} \frac{z^{-k}}{k} \partial_{t_k}. \quad (3)$$

The function F is a dispersionless analogue of the tau-function. The differential equations are obtained by expanding equations (1), (2) in powers of z, ζ . For example, the first two equations of the hierarchy are

$$\begin{cases} 6F_{11}^2 + 3F_{22} - 4F_{13} = 12e^{4F_{00}} \\ 2F_{03} + 4F_{01}^3 + 6F_{01}F_{11} - 6F_{01}F_{02} = 3F_{12}. \end{cases} \quad (4)$$

We use the short-hand notation $F_{mn} \equiv \partial_{t_m} \partial_{t_n} F$.

The dispersionless version of the Pfaff-Toda hierarchy (dPfaff-Toda) [13] is written for a function F of the doubly-infinite set of times $\{\dots, \bar{t}_2, \bar{t}_1, r, s, t_1, t_2, \dots\}$. Since the different hierarchies are never mixed in this paper, we keep the same notation F for the dispersionless tau-function. The real form of the hierarchy, which we will be dealt with, implies that \bar{t}_k is complex conjugate to t_k , s is real and r is purely imaginary. The basic equations are as follows:

$$e^{D(z)D(\zeta)F} \left(1 - \frac{1}{z\zeta} e^{\partial_s(\partial_s+\partial_r+D(z)+D(\zeta))F} \right) = \frac{ze^{-\partial_r D(z)F} - \zeta e^{-\partial_r D(\zeta)F}}{z - \zeta}, \quad (5)$$

$$e^{\bar{D}(\bar{z})\bar{D}(\bar{\zeta})F} \left(1 - \frac{1}{\bar{z}\bar{\zeta}} e^{\partial_s(\partial_s-\partial_r+\bar{D}(\bar{z})+\bar{D}(\bar{\zeta}))F} \right) = \frac{\bar{z}e^{\partial_r \bar{D}(\bar{z})F} - \bar{\zeta}e^{\partial_r \bar{D}(\bar{\zeta})F}}{\bar{z} - \bar{\zeta}}, \quad (6)$$

$$e^{D(z)D(\zeta)F} \left(1 - \frac{1}{z\zeta} e^{\partial_r(\partial_s+\partial_r+D(z)+D(\zeta))F} \right) = \frac{ze^{-\partial_s D(z)F} - \zeta e^{-\partial_s D(\zeta)F}}{z - \zeta}, \quad (7)$$

$$e^{\bar{D}(\bar{z})\bar{D}(\bar{\zeta})F} \left(1 - \frac{1}{\bar{z}\bar{\zeta}} e^{-\partial_r(\partial_s-\partial_r+\bar{D}(\bar{z})+\bar{D}(\bar{\zeta}))F} \right) = \frac{\bar{z}e^{-\partial_s \bar{D}(\bar{z})F} - \bar{\zeta}e^{-\partial_s \bar{D}(\bar{\zeta})F}}{\bar{z} - \bar{\zeta}}, \quad (8)$$

$$e^{-D(z)\bar{D}(\bar{\zeta})F} \left(1 - \frac{1}{z\bar{\zeta}} e^{\partial_r(\partial_r+D(z)-\bar{D}(\bar{\zeta}))F} \right) = 1 - \frac{1}{z\bar{\zeta}} e^{\partial_s(\partial_s+D(z)+\bar{D}(\bar{\zeta}))F}, \quad (9)$$

$$e^{-(\partial_s+\partial_r+D(z))\bar{D}(\bar{\zeta})F} - 1 = \frac{z}{\bar{\zeta}} e^{-\partial_r(\partial_s+D(z)+\bar{D}(\bar{\zeta}))F} \left(e^{-(\partial_s-\partial_r+\bar{D}(\bar{\zeta}))D(z)F} - 1 \right). \quad (10)$$

Here $\bar{D}(\bar{z}) = \sum_{k \geq 1} \frac{\bar{z}^{-k}}{k} \partial_{\bar{t}_k}$ is the complex conjugate counterpart of the differential operator (3). Note that equations (6), (8) are obtained from, respectively, (5), (7) by applying the “bar-operation” $D \rightarrow \bar{D}$, $z \rightarrow \bar{z}$, $\zeta \rightarrow \bar{\zeta}$, $t_k \rightarrow \bar{t}_k$, $s \rightarrow \bar{s} = s$, $r \rightarrow \bar{r} = -r$ which can be treated as complex conjugation provided the function F is real. We see that each equation has a “bar-counterpart”. At the same time, the other two equations, (9) and (10), are real, i.e. they do not change under the complex conjugation. From now on we will not always write explicitly the conjugates of complex equations keeping in mind that they hold simultaneously. In what follows it will be more convenient to introduce the complex conjugate “0th times” $t_0 = s + r$, $\bar{t}_0 = s - r$, so $\partial_{t_0} = \frac{1}{2}(\partial_s + \partial_r)$, $\partial_{\bar{t}_0} = \frac{1}{2}(\partial_s - \partial_r)$.

The differential equations are obtained by expanding (5)–(10) in powers of z , ζ , \bar{z} , $\bar{\zeta}$. The two simplest equations of the hierarchy are

$$\begin{cases} e^{F_{00}} F_{0\bar{1}} = e^{F_{0\bar{0}}} F_{01}, \\ F_{1\bar{1}} = 2 e^{F_{00} + F_{0\bar{0}}} \sinh(2F_{0\bar{0}}). \end{cases} \quad (11)$$

Here $F_{mn} \equiv \partial_{t_m} \partial_{t_n} F$, $F_{m\bar{n}} \equiv \partial_{t_m} \partial_{\bar{t}_n} F$, $F_{\bar{m}\bar{n}} \equiv \partial_{\bar{t}_m} \partial_{\bar{t}_n} F$.

In this work we show that the Pfaff-type hierarchies admit a nice reformulation in terms of elliptic functions (or Jacobi theta functions). After this reformulation, the number of independent equations gets reduced and somewhat unsightly looking equations (1), (2) and especially (5)–(10) assume compact and suggestive forms which look like natural elliptic deformations of the dispersionless KP (or modified KP) and 2DTL hierarchies (see respectively (44), (45) and (46) below in Section 4). Note that in the elliptic parametrization, the modular parameter τ is a dynamical variable. This feature suggests some similarities with the genus 1 Whitham equations [15] and the integrable structures behind boundary value problems in doubly-connected domains in the plane [16].

The elliptic form of the dPfaff-KP hierarchy was obtained in our previous work [17]. This result is reviewed in Section 2. In Section 3 it is extended to the dPfaff-Toda case. In Section 4 we compare the Pfaff-type hierarchies with the more familiar ones.

2 The dispersionless Pfaff-KP hierarchy

Algebraic formulation. Here we deal with the set of real times $\mathbf{t} = \{t_0, t_1, t_2, \dots\}$. In what follows we use the differential operator

$$\nabla(z) = \partial_{t_0} + D(z) \quad (12)$$

which appears to be more convenient than $D(z)$. Introducing the auxiliary functions

$$p(z) = z - \partial_{t_1} \nabla(z) F, \quad w(z) = z^2 e^{-2\partial_{t_0} \nabla(z) F}, \quad (13)$$

we can rewrite equations (1), (2) in a more compact form

$$e^{D(z)D(\zeta)F} \left(1 - \frac{1}{w(z)w(\zeta)} \right) = \frac{p(z) - p(\zeta)}{z - \zeta}, \quad (14)$$

$$e^{-D(z)D(\zeta)F+2\partial_{t_0}^2 F} \frac{w(z) - w(\zeta)}{z - \zeta} = p(z) + p(\zeta). \quad (15)$$

Multiplying them, we get the relation

$$p^2(z) - e^{2F_{00}}(w(z) + w^{-1}(z)) = p^2(\zeta) - e^{2F_{00}}(w(\zeta) + w^{-1}(\zeta))$$

which states that the combination $p^2(z) - e^{2F_{00}}(w(z) + w^{-1}(z))$ does not depend on z . The limit $z \rightarrow \infty$ allows one to express this quantity through derivatives of the function F . As a result we find that $p(z), w(z)$ satisfy the algebraic equation [13]

$$p^2(z) = r^2(w(z) + w^{-1}(z)) - v, \quad (16)$$

where $r = e^{F_{00}}$, $v = 2F_{11} + F_{01}^2 - F_{02}$ are real parameters. This equation defines an elliptic curve, with p, w being algebraic functions on it. The local parameter around ∞ is z^{-1} . As is seen from (13), the functions p and w have respectively a simple and a double pole at infinity.

Elliptic formulation. A natural further step is to uniformize the elliptic curve (16) using the elliptic functions sn , cn , dn or the theta-functions¹ $\theta_a(u) = \theta_a(u|\tau)$ ($a = 1, 2, 3, 4$). It can be done in different ways. Note first that given r, v , the modular parameter $\tau \in \mathbb{H}$ (\mathbb{H} is the upper half-plane) is not uniquely defined because of possible modular transformations. The reality of the coefficients r^2, v implies certain restrictions on possible values of τ . In the standard fundamental domain $\{\tau \in \mathbb{H} \mid |\text{Re } \tau| \leq \frac{1}{2}, |\tau| \geq 1\}$ possible values of τ are as follows: a) $\tau = it, t \geq 1$, b) $\tau = \frac{1}{2} + it, t \geq \sqrt{3}/2$, c) $\tau = e^{i\rho}, \frac{\pi}{3} \leq \rho \leq \frac{2\pi}{3}$.

In what follows we will consider purely imaginary τ (case a)) and choose the uniformization suggested in [17]:

$$w(z) = \frac{\theta_4^2(u(z))}{\theta_1^2(u(z))}, \quad p(z) = \gamma \theta_4^2(0) \frac{\theta_2(u(z)) \theta_3(u(z))}{\theta_1(u(z)) \theta_4(u(z))}, \quad (17)$$

with r, v given by

$$r = \gamma \theta_2(0) \theta_3(0), \quad v = \gamma^2 (\theta_2^4(0) + \theta_3^4(0)). \quad (18)$$

One can check that the equation of the curve becomes equivalent to the identity

$$\theta_4^4(0) \frac{\theta_2^2(u) \theta_3^2(u)}{\theta_1^2(u) \theta_4^2(u)} = \theta_2^2(0) \theta_3^2(0) \left(\frac{\theta_4^2(u)}{\theta_1^2(u)} + \frac{\theta_1^2(u)}{\theta_4^2(u)} \right) - (\theta_2^4(0) + \theta_3^4(0)), \quad (19)$$

which can be proved in the standard way by comparing analytical properties of the both sides. The z -independent factor $\gamma \in \mathbb{R}$ in (17), (18) is, at this stage, an arbitrary parameter. As is shown below, it is a dynamical variable, as well as the modular parameter τ : $\gamma = \gamma(\mathbf{t})$, $\tau = \tau(\mathbf{t})$. The function u in (17) depends on z and on all times: $u(z) = u(z, \mathbf{t})$. Equations (13) show that the functions $w(z), p(z)$ take real values for real z . Taking this

¹Their definition and basic properties are listed in the appendix. Below we will often write simply $\theta_a(u)$ if this does not cause confusion.

into account, it is convenient to normalize $u(z)$ by the condition $u(\infty) = 0$, with the expansion around ∞ being of the form

$$u(z, \mathbf{t}) = \frac{c_1(\mathbf{t})}{z} + \frac{c_2(\mathbf{t})}{z^2} + \dots, \quad c_i \in \mathbb{R}. \quad (20)$$

After the uniformization equations (14) and (15) become identical. Let us take, for example, equation (15) and write it as

$$(z_1^{-1} - z_2^{-1})e^{\nabla_1 \nabla_2 F} = -\frac{w_1 - w_2}{p_1 - p_2} \frac{\mathbf{r}}{\sqrt{w_1 w_2}},$$

where $\nabla_i = \nabla(z_i)$, $p_i = p(z_i)$, etc. The identity

$$\frac{w_1 - w_2}{p_1 + p_2} = -\frac{1}{\gamma \theta_2(0)\theta_3(0)} \frac{\theta_4(u_1)\theta_4(u_2)}{\theta_1(u_1)\theta_1(u_2)} \frac{\theta_1(u_{12})}{\theta_4(u_{12})}$$

(here and below $u_i \equiv u(z_i)$, $u_{ik} \equiv u_i - u_k$) allows us to transform the right hand side to a very simple form:

$$\boxed{(z_1^{-1} - z_2^{-1}) e^{\nabla(z_1) \nabla(z_2) F} = \frac{\theta_1(u(z_1) - u(z_2))}{\theta_4(u(z_1) - u(z_2))}.} \quad (21)$$

This equation encodes the dispersionless Pfaff-KP hierarchy. Note that the limit $z_2 \rightarrow \infty$ in (21) gives the definition of the function $u(z)$ equivalent to the first formula in (17):

$$e^{\partial_{t_0} \nabla(z) F} = z \frac{\theta_1(u(z))}{\theta_4(u(z))}. \quad (22)$$

The $z \rightarrow \infty$ limit of this equation yields $e^{F_{00}} = \mathbf{r} = \pi c_1 \theta_2(0) \theta_3(0)$, hence

$$c_1(\mathbf{t}) = \frac{\gamma(\mathbf{t})}{\pi}, \quad \gamma(\mathbf{t}) = \frac{e^{F_{00}}}{\theta_2(0|\tau) \theta_3(0|\tau)}. \quad (23)$$

In addition, we see from (18) that

$$\frac{\mathbf{v}}{\mathbf{r}^2} = e^{-2F_{00}} (2F_{11} + F_{01}^2 - F_{02}) = \frac{\theta_2^2(0|\tau)}{\theta_3^2(0|\tau)} + \frac{\theta_3^2(0|\tau)}{\theta_2^2(0|\tau)}. \quad (24)$$

This relation makes it clear that the modular parameter τ is expressed through second order partial derivatives of the function F . Due to (23) the same is true for c_1 and γ .

To give yet another instructive form of equation (21), it is convenient to introduce the function

$$S(u|\tau) := \log \frac{\theta_1(u|\tau)}{\theta_4(u|\tau)}. \quad (25)$$

It has the (quasi)periodicity properties $S(u+1|\tau) = S(u|\tau) + i\pi$, $S(u+\tau|\tau) = S(u|\tau)$. The derivative of this function $S'(u) = \partial_u S(u|\tau)$ is given by

$$S'(u) = \pi \theta_4^2(0) \frac{\theta_2(u)\theta_3(u)}{\theta_1(u)\theta_4(u)}. \quad (26)$$

This formula can be easily proved, with the help of identity (56) from the appendix, by comparing analytical properties of the both sides. For the needs of the next section we note here that the identity (19) can be read as a non-linear differential equation for the function S :

$$\left(\frac{S'(u)}{\pi \theta_2(0) \theta_3(0)} \right)^2 = 2 \cosh(2S(u)) - \frac{\theta_2^2(0)}{\theta_3^2(0)} - \frac{\theta_3^2(0)}{\theta_2^2(0)}. \quad (27)$$

Let us take logarithms and apply ∂_{t_0} to both sides of (21). In terms of the function $S(u)$, the equation reads

$$\nabla(z_1) S(u(z_2)|\tau) = \partial_{t_0} S(u(z_1) - u(z_2)|\tau). \quad (28)$$

In particular, this equation means that the left hand side is symmetric with respect to the permutation $z_1 \leftrightarrow z_2$: $\nabla(z_1) S(u(z_2)|\tau) = \nabla(z_2) S(u(z_1)|\tau)$. This symmetry is a manifestation of integrability. In the limit $z_2 \rightarrow \infty$ equation (28) gives:

$$\nabla(z) \log r = \partial_{t_0} S(u(z)|\tau). \quad (29)$$

To connect this with the algebraic formulation, we note that

$$S(u(z)|\tau) = -\frac{1}{2} \log w(z), \quad c_1 S'(u(z)|\tau) = p(z). \quad (30)$$

These formulas directly follow from the definitions and from (26).

3 The dispersionless Pfaff-Toda hierarchy

Algebraic formulation. Now the set of times is $\mathbf{t} = \{\dots, \bar{t}_2, \bar{t}_1, \bar{t}_0, t_0, t_1, t_2, \dots\}$. Accordingly, the operator (12) acquires the “bar-counterpart” $\bar{\nabla}(\bar{z}) = \partial_{\bar{t}_0} + \bar{D}(\bar{z})$. From now on we will work with the times t_0, \bar{t}_0 instead of s, r .

Introducing the auxiliary functions

$$\begin{aligned} P(z) &= z e^{-(\partial_{t_0} + \partial_{\bar{t}_0}) \nabla(z) F}, & W(z) &= z e^{-(\partial_{t_0} - \partial_{\bar{t}_0}) \nabla(z) F}, \\ \bar{P}(z) &= z e^{-(\partial_{t_0} + \partial_{\bar{t}_0}) \bar{\nabla}(z) F}, & \bar{W}(z) &= z e^{(\partial_{t_0} - \partial_{\bar{t}_0}) \bar{\nabla}(z) F}, \end{aligned} \quad (31)$$

we can rewrite equations (5)–(10) in a more compact form

$$\begin{aligned} e^{D(z)D(\zeta)F} \left(1 - \frac{1}{P(z)P(\zeta)} \right) &= \frac{W(z) - W(\zeta)}{z - \zeta} e^{(\partial_{t_0} - \partial_{\bar{t}_0}) \partial_{t_0} F} \\ e^{D(z)D(\zeta)F} \left(1 - \frac{1}{W(z)W(\zeta)} \right) &= \frac{P(z) - P(\zeta)}{z - \zeta} e^{(\partial_{t_0} + \partial_{\bar{t}_0}) \partial_{t_0} F} \\ e^{D(z)\bar{D}(\bar{\zeta})F} \left(1 - \frac{1}{P(z)\bar{P}(\bar{\zeta})} \right) &= 1 - \frac{1}{W(z)\bar{W}(\bar{\zeta})} \\ e^{D(z)\bar{D}(\bar{\zeta})F} \left(W(z) - \bar{W}(\bar{\zeta}) \right) &= \left(P(z) - \bar{P}(\bar{\zeta}) \right) e^{2\partial_{t_0} \partial_{\bar{t}_0} F}, \end{aligned} \quad (32)$$

where $\overline{P(\zeta)} := \bar{P}(\bar{z})$, $\overline{W(\zeta)} := \bar{W}(\bar{z})$. Dividing the first equation by the second one, we get the relation

$$W(z) + W^{-1}(z) - e^{2\partial_{t_0}\partial_{\bar{t}_0}F} (P(z) + P^{-1}(z)) = W(\zeta) + W^{-1}(\zeta) - e^{2\partial_{t_0}\partial_{\bar{t}_0}F} (P(\zeta) + P^{-1}(\zeta))$$

which states that the combination $W(z) + W^{-1}(z) - e^{2\partial_{t_0}\partial_{\bar{t}_0}F} (P(z) + P^{-1}(z)) := C$ does not depend on z . The limit $z \rightarrow \infty$ allows one to express the constant C through derivatives of the function F : $C = 2e^{-(\partial_{t_0}-\partial_{\bar{t}_0})\partial_{t_0}F} \partial_{\bar{t}_0}\partial_{t_1}F$. Dividing the third equation in (32) by the fourth one, we get a relation which states that C is real, i.e., $e^{\partial_{t_0}^2 F} \partial_{t_0}\partial_{\bar{t}_1}F = e^{\partial_{\bar{t}_0}^2 F} \partial_{\bar{t}_0}\partial_{t_1}F$. This is the first equation in (11). As a result, we find that $P(z), W(z)$ satisfy the algebraic equation [13]

$$W(z) + W^{-1}(z) - R^2 (P(z) + P^{-1}(z)) = C, \quad (33)$$

with the real coefficients

$$R^2 = e^{2F_{00}}, \quad C = 2e^{F_{00}-F_{01}} F_{01}. \quad (34)$$

The functions \bar{P}, \bar{W} satisfy the same equation. Like in the case of the dPfaff-KP hierarchy, this equation defines an elliptic curve, with P and W being algebraic functions on it and z^{-1} the local parameter around ∞ . As is seen from (31), both P and W have a simple pole at infinity.

In what follows it is more convenient to work with the functions

$$f(z) = \sqrt{P(z)W(z)} = ze^{-\partial_{t_0}\nabla(z)F}, \quad g(z) = \sqrt{P(z)/W(z)} = e^{-\partial_{\bar{t}_0}\nabla(z)F}. \quad (35)$$

The function f has a simple pole at ∞ while g is regular there. Their complex conjugates are $\overline{f(z)} = \bar{f}(\bar{z}) = \bar{z}e^{-\partial_{\bar{t}_0}\nabla(\bar{z})F}$, $\overline{g(z)} = \bar{g}(\bar{z}) = e^{-\partial_{t_0}\nabla(\bar{z})F}$. In these terms the equation of the elliptic curve reads

$$R^2(f^2g^2 + 1) + Cfg = f^2 + g^2. \quad (36)$$

Note the symmetry $f \leftrightarrow g$. The functions $\bar{f}(z), \bar{g}(z)$ obey the same equation.

Elliptic formulation. The uniformization of the curve (36) in terms of the theta functions $\theta_a(u) = \theta_a(u|\tau)$ can be chosen in the form

$$f(z) = \frac{\theta_4(u(z))}{\theta_1(u(z))}, \quad g(z) = \frac{\theta_4(u(z) + \eta)}{\theta_1(u(z) + \eta)}. \quad (37)$$

Here $u(z) = u(z, \mathbf{t})$ has the same expansion (20) around ∞ but the coefficients are complex and there is also the series $\bar{u}(z) = \bar{u}(\bar{z}) = \bar{u}(\bar{z}, \mathbf{t})$ with conjugate coefficients:

$$u(z, \mathbf{t}) = \frac{c_1(\mathbf{t})}{z} + \frac{c_2(\mathbf{t})}{z^2} + \dots, \quad \bar{u}(z, \mathbf{t}) = \frac{\bar{c}_1(\mathbf{t})}{z} + \frac{\bar{c}_2(\mathbf{t})}{z^2} + \dots. \quad (38)$$

The parameter η is a dynamical variable as well as the modular parameter τ : $\eta = \eta(\mathbf{t})$, $\tau = \tau(\mathbf{t})$. Plugging (37) into the equation of the curve, one can see that it converts into identity if

$$R = \frac{\theta_1(\eta)}{\theta_4(\eta)}, \quad C = 2 \frac{\theta_4^2(0) \theta_2(\eta) \theta_3(\eta)}{\theta_4^2(\eta) \theta_2(0) \theta_3(0)}. \quad (39)$$

We assume that η is real and τ is purely imaginary. This is consistent with reality of R and C .

After the uniformization only two equations in (32) remain independent (say, the first and the third one). Our next task is to represent them in the elliptic form. Let us first rewrite them as

$$(z_1^{-1} - z_2^{-1})e^{\nabla_1 \nabla_2 F} = R^{-1} g_1 g_2 \frac{W_1 - W_2}{1 - P_1 P_2},$$

$$e^{\nabla_1 \bar{\nabla}_2 F} = R^{-1} g_1 \bar{g}_2 \frac{1 - W_1 \bar{W}_2}{1 - P_1 \bar{P}_2},$$

where $\nabla_i = \nabla(z_i)$, $\bar{\nabla}_i = \bar{\nabla}(\bar{z}_i)$, $g_i = g(z_i)$, etc. The identities

$$\begin{aligned} \frac{W_1 - W_2}{1 - P_1 P_2} &= \frac{\theta_1(\eta)}{\theta_4(\eta)} \frac{\theta_1(u_1 + \eta)}{\theta_4(u_1 + \eta)} \frac{\theta_1(u_2 + \eta)}{\theta_4(u_2 + \eta)} \cdot \frac{\theta_1(u_1 - u_2)}{\theta_4(u_1 - u_2)}, \\ \frac{1 - W_1 \bar{W}_2}{1 - P_1 \bar{P}_2} &= \frac{\theta_1(\eta)}{\theta_4(\eta)} \frac{\theta_1(u_1 + \eta)}{\theta_4(u_1 + \eta)} \frac{\theta_1(\bar{u}_2 + \eta)}{\theta_4(\bar{u}_2 + \eta)} \cdot \frac{\theta_1(u_1 + \bar{u}_2 + \eta)}{\theta_4(u_1 + \bar{u}_2 + \eta)} \end{aligned}$$

allow one to represent the equations in the form

$$\begin{aligned} (z_1^{-1} - z_2^{-1}) e^{\nabla(z_1) \nabla(z_2) F} &= \frac{\theta_1(u(z_1) - u(z_2))}{\theta_4(u(z_1) - u(z_2))} \\ e^{\nabla(z_1) \bar{\nabla}(\bar{z}_2) F} &= \frac{\theta_1(u(z_1) + \bar{u}(z_2) + \eta)}{\theta_4(u(z_1) + \bar{u}(z_2) + \eta)} \\ (z_1^{-1} - z_2^{-1}) e^{\bar{\nabla}(z_1) \bar{\nabla}(z_2) F} &= \frac{\theta_1(\bar{u}(z_1) - \bar{u}(z_2))}{\theta_4(\bar{u}(z_1) - \bar{u}(z_2))} \end{aligned}$$

(40)

The first equation is the same as (21). This means that a “half” of the dispersionless Pfaff-Toda hierarchy (with fixed bar-times) coincides with the Pfaff-KP one. This fact can not be so transparently seen in the algebraic formulation. The third equation is the bar-version of the first one. It represents another copy of the dPfaff-KP hierarchy, now with respect to the bar-times \bar{t}_k with fixed t_k 's. The second equation contains mixed derivatives with respect to the times $\{t_k\}$ and $\{\bar{t}_k\}$ and thus it couples the two hierarchies into the more general one. This equation is invariant under complex conjugation.

The $z_2 \rightarrow \infty$ limits of equations (40) yield:

$$e^{\partial_{t_0} \nabla(z) F} = z \frac{\theta_1(u(z))}{\theta_4(u(z))}, \quad e^{\partial_{\bar{t}_0} \nabla(z) F} = \frac{\theta_1(u(z) + \eta)}{\theta_4(u(z) + \eta)}, \quad (41)$$

which are nothing else than the expressions for the functions f and g (37) combined with their definition (35). The further $z \rightarrow \infty$ expansion of these relations gives $e^{F_{00}} = \pi c_1 \theta_2(0) \theta_3(0)$ from the leading terms of the first one and $F_{01} = c_1 S'(\eta)$ from the $O(z^{-1})$ terms of the second one (the function S is defined in (25)). From (39) it follows that $R = e^{S(\eta)}$, $C/R = \frac{2S'(\eta)}{\pi \theta_2(0) \theta_3(0)}$. We can use (27) with the substitution $u \rightarrow \eta$ to get

$$R^2 + R^{-2} \left(1 - \frac{C^2}{4}\right) = 2 \cosh(2F_{00}) - e^{-F_{00} - F_{00}} F_{01} F_{0\bar{1}} = \frac{\theta_2^2(0|\tau)}{\theta_3^2(0|\tau)} + \frac{\theta_3^2(0|\tau)}{\theta_2^2(0|\tau)}. \quad (42)$$

Similarly to (24), this equation means that the modular parameter τ is expressed neatly through second order partial derivatives of the function F . The same is true for c_1 and η .

Equations (40) imply the following relations:

$$\begin{aligned}\nabla(z_1)S(u(z_2)) &= \partial_{t_0}S(u(z_1)-u(z_2)), \quad \nabla(z_1)S(u(z_2)+\eta) = \partial_{\bar{t}_0}S(u(z_1)-u(z_2)), \\ \bar{\nabla}(\bar{z}_1)S(u(z_2)) &= \partial_{t_0}S(\bar{u}(\bar{z}_1)+u(z_2)+\eta), \quad \bar{\nabla}(\bar{z}_1)S(u(z_2)+\eta) = \partial_{\bar{t}_0}S(\bar{u}(\bar{z}_1)+u(z_2)+\eta).\end{aligned}\tag{43}$$

In particular, we have $\nabla(z) \log R = \partial_{\bar{t}_0}S(u(z)) = \partial_{t_0}S(u(z) + \eta)$.

4 Comparison with other hierarchies

It is instructive to compare the dispersionless Pfaff-type hierarchies with the more familiar dispersionless KP (dKP), mKP (dmKP) and 2DTL (d2DTL) ones:

$$\text{dKP:} \quad e^{D(z)D(\zeta)F} = 1 - \frac{\partial_{t_1}(D(z)-D(\zeta))F}{z-\zeta}, \tag{44}$$

$$\text{dmKP:} \quad e^{D(z)D(\zeta)F} = \frac{ze^{-\partial_{t_0}D(z)F} - \zeta e^{-\partial_{t_0}D(\zeta)F}}{z-\zeta}, \tag{45}$$

$$\text{d2DTL:} \quad \begin{cases} e^{D(z)D(\zeta)F} = \frac{ze^{-\partial_{t_0}D(z)F} - \zeta e^{-\partial_{t_0}D(\zeta)F}}{z-\zeta} \\ e^{-D(z)\bar{D}(\bar{\zeta})F} = 1 - (z\bar{\zeta})^{-1}e^{\partial_{t_0}(\partial_{t_0}+D(z)+\bar{D}(\bar{\zeta}))F}. \end{cases} \tag{46}$$

In the dKP case the (real) times are $\{t_1, t_2, \dots\}$. In the dmKP case this set of times is supplemented by (real) t_0 . In the d2DTL case the time t_0 is real while the other ones are complex, i.e. we have two sets of times $\{t_1, t_2, \dots\}$ and $\{\bar{t}_1, \bar{t}_2, \dots\}$ which are complex conjugate to each other. Note that the first equation in (46) and equation (45) are identical. For more details see [18, 19].

dPfaff-KP versus dKP and dmKP. First of all, let us note that the dmKP equation (45) implies (44). Indeed, writing (45) in the form $z_{12}e^{D_1D_2F} = z_1e^{-D_1F_0} - z_2e^{-D_2F_0}$ and summing such equations for the pairs 12, 23, 31, we get $z_{12}e^{D_1D_2F} + z_{23}e^{D_2D_3F} + z_{31}e^{D_1D_3F} = 0$, and tending $z_3 \rightarrow \infty$ here, we arrive at (44). In a similar way, one can show that equation (45), written through the operator $\nabla(z)$ in the form $(z_1^{-1} - z_2^{-1})e^{\nabla_1\nabla_2F} = z_1^{-1}e^{-\nabla_1F_0} - z_2^{-1}e^{-\nabla_2F_0}$, implies the equation

$$\begin{vmatrix} 1 & z_1^{-1} & e^{\nabla_2\nabla_3F} \\ 1 & z_2^{-1} & e^{\nabla_1\nabla_3F} \\ 1 & z_3^{-1} & e^{\nabla_1\nabla_2F} \end{vmatrix} = 0. \tag{47}$$

In its turn, this equation implies similar antisymmetric determinant relations containing more points. In particular, it is an easy algebraic exercise to show that it follows from (47) that

$$\begin{vmatrix} 1 & z_1^{-1} & z_1^{-2} & e^{(\nabla_2 \nabla_3 + \nabla_3 \nabla_4 + \nabla_4 \nabla_2)F} \\ 1 & z_2^{-1} & z_2^{-2} & e^{(\nabla_1 \nabla_3 + \nabla_3 \nabla_4 + \nabla_4 \nabla_1)F} \\ 1 & z_3^{-1} & z_3^{-2} & e^{(\nabla_1 \nabla_2 + \nabla_2 \nabla_4 + \nabla_4 \nabla_1)F} \\ 1 & z_4^{-1} & z_4^{-2} & e^{(\nabla_1 \nabla_2 + \nabla_2 \nabla_3 + \nabla_3 \nabla_1)F} \end{vmatrix} = 0. \quad (48)$$

In fact this is the dispersionless limit of one of the higher equations of the difference Hirota hierarchy [20].

Now let us turn to the dPfaff-KP hierarchy in the elliptic form. Plugging the left hand side of equation (21) (for different pairs of variables) into the identity

$$\begin{aligned} & \frac{\theta_1(u_{12})\theta_1(u_{23})\theta_1(u_{31})}{\theta_4(u_{12})\theta_4(u_{23})\theta_4(u_{31})} - \frac{\theta_1(u_{12})\theta_1(u_{24})\theta_1(u_{41})}{\theta_4(u_{12})\theta_4(u_{24})\theta_4(u_{41})} \\ & + \frac{\theta_1(u_{13})\theta_1(u_{34})\theta_1(u_{41})}{\theta_4(u_{13})\theta_4(u_{34})\theta_4(u_{41})} - \frac{\theta_1(u_{23})\theta_1(u_{34})\theta_1(u_{42})}{\theta_4(u_{23})\theta_4(u_{34})\theta_4(u_{42})} = 0, \end{aligned} \quad (49)$$

we get the determinant relation for the function F of *precisely the same form (48) as the higher equation of the dmKP hierarchy*. At the same time, none of solutions to the latter obey equations (1), (2) of the dPfaff-KP hierarchy. Indeed, as is shown above, equation (44) is valid for the dmKP hierarchy; plugging it into (1), we get $e^{2\partial_{t_0}(2\partial_{t_0}+D(z)+D(\zeta))F} = 0$ which is impossible for any F . For example, the simplest solution to (47) and (48) is $F = 0$ which is not a solution to (1). We see that (47) implies (48) but not vice versa.

Dispersionless Toda chain. Let us also mention the familiar reduction of the d2DTL hierarchy obtained by imposing the conditions $\partial_{t_k} F = \partial_{\bar{t}_k} F$ for all $k \geq 1$ which means that $D(z)F = \bar{D}(\zeta)F$. This hierarchy is called the dispersionless Toda chain (dTC):

$$\text{dTC:} \quad \begin{cases} e^{D(z)D(\zeta)F} = \frac{ze^{-\partial_{t_0}D(z)F} - \zeta e^{-\partial_{t_0}D(\zeta)F}}{z - \zeta} \\ e^{-D(z)D(\zeta)F} = 1 - (z\zeta)^{-1} e^{\partial_{t_0}(\partial_{t_0}+D(z)+D(\zeta))F}. \end{cases} \quad (50)$$

In terms of the function $\omega(z) = ze^{-\frac{1}{2}F_{00}-\partial_{t_0}D(z)F}$, we can rewrite (50) as

$$\begin{cases} e^{-\frac{1}{2}F_{00}+D(z)D(\zeta)F} = \frac{\omega(z) - \omega(\zeta)}{z - \zeta} \\ e^{-D(z)D(\zeta)F} = 1 - \frac{1}{\omega(z)\omega(\zeta)}. \end{cases} \quad (51)$$

Note the similarity with (14), (15). Following [10], we multiply these equations to conclude that the combination $z - e^{\frac{1}{2}F_{00}}(\omega(z) + \omega^{-1}(z))$ does not depend on z . Tending

$z \rightarrow \infty$ we find this constant to be equal to F_{01} . Therefore, the variables z and ω satisfy the algebraic equation

$$z = e^{\frac{1}{2}F_{00}} \left(\omega(z) + \frac{1}{\omega(z)} \right) + F_{01}, \quad (52)$$

which defines a rational (genus 0) curve.

Finally, we shall show that the dPfaff-KP hierarchy (1), (2) contains the dispersionless Toda chain as a reduction (see [10, Proposition 4.1]). Consider solutions to the dPfaff-KP hierarchy such that $\partial_{t_{2k+1}} F = 0$ for all $k \geq 0$. Redefine the times as follows: $\tilde{t}_n = 2t_{2n}$, $n \geq 1$, $\tilde{t}_0 = \frac{1}{2}t_0$. Then it is easy to check that equations (1), (2) convert into the system

$$\begin{cases} e^{\tilde{D}(z^2)\tilde{D}(\zeta^2)\tilde{F}} = \frac{ze^{-\partial_{\tilde{t}_0}\tilde{D}(z^2)\tilde{F}} - \zeta e^{-\partial_{\tilde{t}_0}\tilde{D}(\zeta^2)\tilde{F}}}{z^2 - \zeta^2} \\ e^{-\tilde{D}(z^2)\tilde{D}(\zeta^2)\tilde{F}} = 1 - (z^2\zeta^2)^{-1}e^{\partial_{\tilde{t}_0}(\partial_{\tilde{t}_0} + \tilde{D}(z^2) + \tilde{D}(\zeta^2))\tilde{F}} \end{cases} \quad (53)$$

for the function $\tilde{F}(\tilde{t}_0, \tilde{t}_1, \tilde{t}_2, \dots) = F(t_0, 0, t_2, 0, t_4, 0, \dots)$, where $\tilde{D}(z) = \sum_{k \geq 1} \frac{z^{-k}}{k} \frac{\partial}{\partial \tilde{t}_k}$, which is equivalent to (50).

Appendix

The Jacobi's theta functions $\theta_a(u) = \theta_a(u|\tau)$, $a = 1, 2, 3, 4$, are defined by the formulas

$$\begin{aligned} \theta_1(u) &= -\sum_{k \in \mathbb{Z}} \exp \left(\pi i \tau \left(k + \frac{1}{2} \right)^2 + 2\pi i \left(u + \frac{1}{2} \right) \left(k + \frac{1}{2} \right) \right), \\ \theta_2(u) &= \sum_{k \in \mathbb{Z}} \exp \left(\pi i \tau \left(k + \frac{1}{2} \right)^2 + 2\pi i u \left(k + \frac{1}{2} \right) \right), \\ \theta_3(u) &= \sum_{k \in \mathbb{Z}} \exp \left(\pi i \tau k^2 + 2\pi i u k \right), \\ \theta_4(u) &= \sum_{k \in \mathbb{Z}} \exp \left(\pi i \tau k^2 + 2\pi i \left(u + \frac{1}{2} \right) k \right), \end{aligned} \quad (54)$$

where the modular parameter τ is such that $\text{Im } \tau > 0$. The function $\theta_1(u)$ is odd, the other three functions are even. It is convenient to understand the index a modulo 4, i.e., to identify $\theta_a(z) \equiv \theta_{a+4}(z)$. Set $\omega_0 = 0$, $\omega_1 = \frac{1}{2}$, $\omega_2 = \frac{1+\tau}{2}$, $\omega_3 = \frac{\tau}{2}$ then the function $\theta_a(u)$ has simple zeros at the points of the lattice $\omega_{a-1} + \mathbb{Z} + \mathbb{Z}\tau$. The theta functions have the following quasi-periodic properties under shifts by 1 and τ :

$$\begin{aligned} \theta_a(u+1) &= e^{\pi i(1+2\partial_\tau \omega_{a-1})} \theta_a(u), \\ \theta_a(u+\tau) &= e^{\pi i(a+2\partial_\tau \omega_{a-1})} e^{-\pi i \tau - 2\pi i u} \theta_a(u). \end{aligned} \quad (55)$$

Shifts by the half-periods relate the different theta functions to each other. We also mention the identity

$$\theta'_1(0) = \pi \theta_2(0) \theta_3(0) \theta_4(0). \quad (56)$$

Many useful formulas with the theta functions can be found in [21].

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